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Some density properties of the closed unit ball of L_1

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ABSTRACT

In this paper we present some properties on the densifiability of certain subsets of L_1. In particular, we prove that the closed unit ball B_1 in L_1 can be filled by weak curves with arbitrarily small density in the same way as it is done in the unit ball of a finite-dimensional space, but a very peculiar detail. Some weak density properties of the set $\mathcal{D}$, of all probability functions of L_1, are used to approach a solution of an infinite-dimensional global optimization problem, posed on $\mathcal{D}$, by means of solutions of one-dimensional global optimization problems posed on each curve that densifies $\mathcal{D}$.

1. Introduction

It is well known [1,5,7] that in the space $L_p[0,1]$ (briefly L_p), $p \geqslant 1$, of all real Lebesgue-measurable functions f on $I = [0,1]$ of pth power integrable, the formula

$$\|f\|_p \equiv \left(\int_0^1 |f(x)|^p \, dx \right)^{\frac{1}{p}}$$

defines a norm if no distinction is made between equivalent functions on $[0,1]$, where the meaning of equivalence is taken as equality almost everywhere (a.e.). Furthermore, $(L_p, \| \, \|_p)$ is a Banach for this topology, also called the strong topology to distinguish that from the weak topology which, from now on, will be denoted by ω_p. In the present paper we will take $p = 1$, so the reference space where we will develop our research and conclusions will be L_1.

As curves are crucial to introduce in L_1 the concept of densifiability, it is convenient, firstly, to precise that the notion of curve will be taken in a wide sense, that is, if E is a topological space, any continuous mapping on the unit interval

$$\gamma : I \to E$$

will be considered as a curve in E.

Since in a Banach space E of finite dimension the closed unit ball B_E is a compact and convex set, it is densifiable, that is, B_E has the property of being "filled" by α-dense curves γ_α for arbitrary $\alpha > 0$. It means that there exist curves γ_α, with images γ_α^* contained in B_E, such that the Hausdorff distance [4]

$$d_{\mathcal{H}}\left(\gamma_\alpha^*, B_E\right) \leqslant \alpha, \quad \alpha > 0. \tag{1.1}$$

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Moreover, B_E is a Peano continuum, i.e., it is a compact, connected and locally connected set. Then, by the Hahn–Mazurkiewicz theorem [3], there exists a curve Γ whose image $\Gamma^* \equiv B_E$ and, consequently,

$$d_{\mathcal{H}}(\Gamma^*, B_E) = 0.$$

Hence, in a normed space E of finite dimension, the closed unit ball B_E is a densifiable set and, in particular, it is the image of a Peano curve or a space-filling curve.

In [6] we saw that the property of densifiability of the closed unit ball is a characteristic of finite-dimensional spaces. There it was explicitly showed:

"The closed unit ball B_E of a Banach space E is densifiable if and only if E has finite dimension".

Therefore, the closed unit ball B_1 of L_1 is not densifiable for the topology of the norm. Nevertheless, as we will see below, the unit ball B_1 of L_1 has for the weak topology a similar behaviour to that of the unit ball in finite-dimensional spaces. In fact, we will prove that B_1 is a densifiable set for that topology. To prove it, firstly, since the space (L_1, ω_1) is not metrizable [5], we will need to substitute the notion of α-density, expressed in (1.1), by another concept which does not require the existence of a metric. This concept is that of V-density, V a weak 0-neighborhood, which will be used below to obtain our purposes.

Finally, as a consequence of the densifiability of certain subsets, say A, of L_1 we can contemplate this property as a tool such that the Peano continua defined by the images of the α-dense curves that densify A approach a limit: A itself. Thus, a multivariable global optimization problem can be reduced to a univariable one in the following sense:

Let A be a densifiable set of L_1 and f a real continuous function on A. The problem

$$\inf\{f(x)\colon\ x \in A\} \tag{1.2}$$

can be approximated by

$$\min\{f(\gamma_\alpha(t))\colon\ t \in I\} \tag{1.3}$$

where γ_α are α-dense curves in A and $\alpha \to 0$. Following this idea, we will exhibit an algorithm to solve a global optimization problem of type (1.2) in the set $\mathcal{D}$ of L_1 defined by

$$\mathcal{D} \equiv \left\{f \in L_1\colon\ f \geqslant 0 \text{ and } \int_{[0,1]} f = 1\right\}. \tag{1.4}$$

2. Alpha-dense and V-dense curves. Densifiable sets

An equivalent form to that of (1.1) to express the α-density is given by means of the following definition.

Definition 1. In a metric space (E, d), given a subset A and a non-negative real α, a curve

$$\gamma : I \to E$$

is said to be α-dense in A if and only if

$$\gamma(I) \subset A,$$

and for any $x \in A$ there exists $t \in I$ such that

$$d(x, \gamma(t)) \leqslant \alpha.$$

Observe that if a curve γ densifies a compact set A with density α then γ also densifies A with density α' for all $\alpha' \geqslant \alpha$. It is easy to check that the minimal α satisfying the above definition coincides with the Hausdorff distance

$$d_{\mathcal{H}}(\gamma^*, A),$$

where γ^* is the image $\gamma(I)$.

In particular, if a curve γ densifies a compact A with density $\alpha = 0$, necessarily the image $\gamma^* \equiv A$, and then γ is called a Peano curve or a space-filling curve.

Definition 2. A subset A of a metric space (E, d) is said to be densifiable if for any $\alpha > 0$ there exists an α-dense curve in A.

For instance, it is easy to see that in the euclidean space R^N, $N \geqslant 1$, any cube $K = \prod_{i=1}^N [a_i, b_i]$ is densifiable; in [2] there is an explicit construction of α-dense curves in K, even of class $\mathcal{C}^\infty$, with arbitrarily small α. Furthermore, because of any bounded closed cube is a Peano continuum, from the mentioned Hahn–Mazurkiewicz theorem, K can be also considered as the image of a Peano curve.

Clearly, in a metric space (E,d), the class of all densifiable sets, $\mathcal{D}_E$, contains the class of all Peano continua, $\mathcal{P}_E$, and $\mathcal{D}_E$ is in turn contained in the class of all precompact and connected subsets, $\mathcal{PC}_E$. That is

$$\mathcal{P}_E \subset \mathcal{D}_E \subset \mathcal{PC}_E.$$

However the three classes are distinct such as the following example shows:

In the euclidean plane the set

$$K_1 = \left\{\left(x, \sin\frac{1}{x}\right):\ 0 < x \leqslant 1\right\} \cup \left\{(0, y):\ -1 \leqslant y \leqslant 1\right\}$$

is densifiable but it is not a Peano continuum, and the set

$$K_2 = \left\{\left(x, \sin\frac{1}{x}\right):\ x \in [-1, 0) \cup (0, 1]\right\} \cup \left\{(0, y):\ -1 \leqslant y \leqslant 1\right\}$$

is precompact and connected but it is not densifiable.

Whenever E is a topological vector space, not necessarily metrizable, the concept of α-density is substituted by that of V-density, V being a given 0-neighborhood.

Definition 3. Given a subset A in a topological vector space $(E,\mathcal{T})$ and V a 0-neighborhood, a curve $\gamma : I \to E$ is said to be V-dense in A if and only if

$$\gamma(I) \subset A,$$

and for any $x \in A$ there exists $t \in I$ such that

$$x - \gamma(t) \in V.$$

Definition 4. A subset A of a topological vector space $(E,\mathcal{T})$ is said to be densifiable if for any 0-neighborhood V there exists a V-dense curve in A.

A large class of densifiable sets in topological vector spaces is supplied by means of the following result.

Proposition 5. *In a locally convex space $(E,\mathcal{T})$ any bounded convex set A is densifiable for the weak topology $\sigma(E,E')$ (i.e. A is densifiable in E endowed with the weak topology $\sigma(E,E')$).*

Proof. The boundedness of A implies that A is $\sigma(E,E')$ precompact [7]. Then, given an arbitrary weak neighborhood of 0, V, there exists a finite subset

$$S = \{f_1, \ldots, f_n\} \subset A$$

such that

$$A \subset S + V.$$

Now, because of the convexity, by joining the points of S by means of line segments it obtains a polygonal contained in A that determines a curve γ_S whose image γ_S^* satisfies

$$A \subset \gamma_S^* + V.$$

Consequently, the proposition follows. □

Corollary 6. *The closed unit ball B_E of a normed space E is weakly densifiable.*

Corollary 7. *The set*

$$\mathcal{D} \equiv \left\{ f \in L_1:\ f \geqslant 0 \text{ and } \int_{[0,1]} f = 1 \right\}$$

is weakly densifiable.

Definition 8. Let A be a subset of a normed space E and, for each $\alpha \geqslant 0$, $\mathcal{C}_\alpha^n(A)$, the class of all α-dense curves in A for the topology of the norm. We define the degree of densifiability in norm of A, denoted by δ_A^n, as

$$\delta_A^n = \begin{cases} \infty, & \text{if } \mathcal{C}_\alpha^n(A) = \emptyset \text{ for all } \alpha, \\ \inf\{\alpha \geqslant 0:\ \mathcal{C}_\alpha^n(A) \neq \emptyset\}, & \text{otherwise.} \end{cases}$$

Obviously, if A is unbounded then the class $\mathcal{C}_\alpha^n(A)$ is empty for all $\alpha \geqslant 0$ and, consequently, $\delta_A^n = \infty$.

In [6], there is computed the degree of densifiability of the closed unit ball B_E of a Banach space E

$$\delta_{B_E}^n = \begin{cases} 0 & \text{if } E \text{ is of finite dimension,} \\ 1 & \text{if } E \text{ is of infinite dimension.} \end{cases}$$

Now, if we consider a normed space E endowed with the weak topology $\sigma(E, E')$, we learn that given a weak neighborhood of 0, V, there exists some $\beta > 0$ such that

$$\beta B_E \subset V.$$

The number β will allow us to define the important concept of spherical order of a weak neighborhood of 0.

Definition 9. Let E be a normed space, B_E its closed unit ball, and V an absolutely convex closed weak proper neighborhood of 0. Then the spherical order of V, noted λ_V, is defined as

$$\lambda_V \equiv \sup\{\beta > 0\colon\ \beta B_E \subset V\}. \tag{2.1}$$

The next result establishes an easy formula to find the spherical order of each weak neighborhood of the subbasis of L_1 defined by the polars of the elements of the dual space L_∞.

Proposition 10. *For each $g \in L_\infty$, $g \neq 0$, the polar $\{g\}^\circ$, is a weak neighborhood of 0 in L_1 of spherical order*

$$\lambda_{V_g} = \frac{1}{\|g\|_\infty}.$$

Proof. Let us denote by V_g the polar $\{g\}^\circ$. Then, we claim that for any β belonging to the interval $[0, \frac{1}{\|g\|_\infty}]$ one has

$$\beta B_1 \subset V_g, \tag{2.2}$$

where B_1 denotes the closed unit ball in L_1. Indeed, for $\beta = 0$, (2.2) is immediate, so assume $\beta > 0$. Then, if (2.2) is not true, there exists some $f \in B_1$ such that

$$\left|\int_I (fg)\right| > \frac{1}{\beta} \geqslant \|g\|_\infty. \tag{2.3}$$

Since $\|f\|_1 \leqslant 1$, the Hölder inequality [1,5,7]

$$\|f\|_1 \|g\|_\infty \geqslant \int_I \big(|fg|\big)$$

and (2.3) lead to the contradiction

$$\|g\|_\infty \geqslant \|g\|_\infty \|f\|_1 \geqslant \int_I \big(|fg|\big) \geqslant \left|\int_I (fg)\right| > \frac{1}{\beta} \geqslant \|g\|_\infty,$$

which proves (2.2), as claimed. Now taking into account (2.2) and the definition of spherical order, we have

$$\lambda_{V_g} \geqslant \frac{1}{\|g\|_\infty}. \tag{2.4}$$

Suppose

$$\lambda_{V_g} > \frac{1}{\|g\|_\infty}.$$

Then, there exists β such that

$$\lambda_{V_g} > \beta > \frac{1}{\|g\|_\infty}$$

satisfying

$$\beta B_1 \subset V_g.$$

In consequence,

$$\left|\int_I (fg)\right| \leqslant \frac{1}{\beta} < \|g\|_\infty, \quad \text{for all } f \in B_1. \tag{2.5}$$

For sufficiently large n determine a Borel set $A_n \subset [0,1]$ with Lebesgue measure $0 < m(A_n) \leqslant 1$ such that

$$\bigl|\bigl|g(x)\bigr| - \|g\|_\infty\bigr| < \frac{1}{n}, \quad \text{for all } x \in A_n, \tag{2.6}$$

where, either $g(x) > 0$ or $g(x) < 0$ for all $x \in A_n$. Now, define the function

$$f_n = \frac{1}{m(A_n)} \chi_{A_n},$$

where χ_{A_n} denotes the characteristic function of A_n. Since $f_n \in B_1$ for all n, by substituting in (2.5) and assuming, for instance that $g(x) > 0$, we have

$$\left|\int_I (f_n g)\right| = \frac{1}{m(A_n)} \int_{A_n} g \leqslant \frac{1}{\beta} < \|g\|_\infty. \tag{2.7}$$

According to (2.6), we can express

$$\int_{A_n} g = m(A_n)\|g\|_\infty + \delta_n$$

with

$$|\delta_n| < \frac{1}{n} m(A_n). \tag{2.8}$$

Thus (2.7) becomes

$$\left|\int_I (f_n g)\right| = \|g\|_\infty + \frac{1}{m(A_n)}\delta_n \leqslant \frac{1}{\beta} < \|g\|_\infty.$$

Hence, from (2.8), by taking the limit we get

$$\lim_{n\to\infty}\left|\int_I (f_n g)\right| = \|g\|_\infty \leqslant \frac{1}{\beta} < \|g\|_\infty,$$

which is a contradiction. In consequence, the equality occurs in (2.4) and then the result follows. □

There is no difficulty to extend the foregoing proposition to a basis of weak neighborhoods of 0.

Corollary 11. *In L_1, the spherical order of any weak 0-neighborhood of the form*

$$V \equiv \bigl\{f \in L_1 \colon \bigl|\langle f, g_i\rangle\bigr| \leqslant 1 \colon 1 \leqslant i \leqslant n\bigr\} \quad (g_i \in L_\infty)$$

is given by

$$\lambda_V = \frac{1}{\max_{1\leqslant i\leqslant n}\{\|g_i\|_\infty\}}.$$

3. The densifiability of the closed unit ball of L_1

Now, we define the degree of weak densifiability of a subset of a normed space E similarly as we have defined the degree of densifiability in norm.

Definition 12. Let A be a subset of a normed space E and, for each V absolutely convex closed weak proper neighborhood of 0, $\mathcal{C}_V^\sigma(A)$, the class of all V-dense curves in A for the weak topology. We define the degree of weak densifiability, δ_A^σ, of A as

$$\delta_A^\sigma = \begin{cases} \infty, & \text{if } \mathcal{C}_V^\sigma(A) = \emptyset \text{ for all } V \in \mathcal{V}, \\ \inf\{\lambda_V \colon V \in \mathcal{V} \text{ with } \mathcal{C}_V^\sigma(A) \neq \emptyset\}, & \text{otherwise}, \end{cases}$$

where λ_V is the spherical order of V defined in (2.1) and $\mathcal{V}$ is the family of all absolutely convex closed weak proper neighborhood of 0.

Since L_1 is infinite-dimensional, from [6], the degree of densifiability in norm of its closed unit ball B_1 is 1. However, in the next result we will show that the degree of weak densifiability of B_1 is 0.

Theorem 13. *In the space L_1 the closed unit ball B_1 has degree of weak densifiability* 0.

Proof. From Corollary 6, B_1 is weakly densifiable and then for any weak neighborhood of 0, say V, the class $\mathcal{C}_V^\sigma(B_1)$ is non-empty. Hence the degree of densifiability $\delta_{B_1}^\sigma$ is finite. It only remains to prove that there exist weak 0-neighborhoods V with spherical orders, λ_V, arbitrarily small. Indeed, consider for any $\lambda > 0$ the sets defined by

$$V_\lambda \equiv \lambda\{h\}^\circ, \tag{3.1}$$

where h is the function identically equal to 1 on $[0,1]$. Now, it is immediate to check that (3.1) defines the weak 0-neighborhoods desired. □

Observe that the above theorem implies that the closed unit ball of L_1, endowed with the weak topology, has the same property as that of the closed unit ball of any finite-dimensional space. It means that in the space L_1 the closed unit ball B_1 can be densified by weakly Peano continua with arbitrarily small density. However, there exists a fundamental difference, namely:

In any metric space, if a curve γ is α-dense, then γ is also α'-dense for all $\alpha' > \alpha$. Nevertheless, in L_1, if a curve γ is V-dense in B_1 with V a weak neighborhood of 0 of spherical order λ, then γ is not necessarily V'-dense for all weak neighborhood of 0, V', of spherical order $\lambda' > \lambda$.

This irregular property of the closed unit ball of L_1 will be demonstrated in the next result.

Theorem 14. *Let γ be an arbitrary curve in the closed unit ball B_1 of L_1 endowed with the weak topology. Let $0 < \lambda < 1$. Then there exists a weak 0-neighborhood V of spherical order λ such that the curve γ is not V-dense in B_1.*

Proof. Firstly, a subset F of L_1 is said to be uniformly integrable [1] if and only if

$$\sup_{f\in F} \int_{\{|f|>a\}} |f(t)|\,dt \to 0 \quad \text{as } a\to\infty.$$

Then the set

$$G \equiv \{f_n = n\chi_{[0,\frac{1}{n}]},\ n\in\mathbb{N}\},$$

$\chi_{[0,\frac{1}{n}]}$ being the characteristic function of $[0,\frac{1}{n}]$ for $n\in\mathbb{N}$, has the property that any infinite subset G' of G is not uniformly integrable since for each n

$$\sup_{f\in G'} \int_{\{|f|>n\}} |f(t)|\,dt = 1.$$

Since the image of γ, γ^*, is weakly compact, from [1, Thm. 13.6], it is uniformly integrable and then the set

$$\gamma^*\cap G$$

is finite. Therefore, there exists

$$n_0 \geqslant 0 \quad \text{such that} \quad f_n \notin \gamma^* \quad \text{for all } n > n_0,\ \text{with } f_n\in G. \tag{3.2}$$

On the other hand, the uniform integrability of γ^* [1, Prop. 13.1] implies that for every $\epsilon > 0$ there exists $\delta > 0$ such that

$$\int_A |f(t)|\,dt < \epsilon \tag{3.3}$$

for all $f\in\gamma^*$ and for all Borel sets $A\subset[0,1]$ with Lebesgue measure $m(A) < \delta$.

Now, given an arbitrary λ in the interval $(0,1)$ let ϵ be such that

$$0 < \epsilon < 1-\lambda.$$

Then, from (3.3), we determine the corresponding δ and, because of (3.2), we select an integer

$$m > \max\left\{n_0, \frac{1}{\delta}\right\}. \tag{3.4}$$

Now, we define the function

$$g_m \equiv \chi_{[0,\frac{1}{m}]},$$

where $\chi_{[0,\frac{1}{m}]}$ denotes the characteristic function of $[0,\frac{1}{m}]$. Thus we claim that the set

$$V \equiv \lambda\{g_m\}^\circ$$

is a weak 0-neighborhood of spherical order λ such that the curve γ is not V-dense in B_1. Indeed, firstly, we express

$$V = \{h_m\}^\circ,$$

with

$$h_m \equiv \frac{1}{\lambda} g_m$$

and norm

$$\|h_m\|_\infty = \frac{1}{\lambda}.$$

Then, by applying Proposition 10, V is a weak 0-neighborhood of spherical order λ. Now, by assuming that the curve γ is V-dense in B_1, it follows that

$$B_1 \subset \gamma^* + V.$$

Thus, given the function $f_m \in G \subset B_1$ with m satisfying (3.4), there exists some function $l_m \in \gamma^*$ such that

$$f_m - l_m \in V$$

or equivalently

$$\left| \int_{[0,1]} \big((f_m - l_m) g_m\big) \right| \leqslant \lambda. \tag{3.5}$$

However, for arbitrary $f \in \gamma^*$ one has

$$\left| \int_{[0,1]} \big((f_m - f) g_m\big) \right| = \left| \int_{[0,\frac{1}{m}]} (f_m - f) \right| = \left| 1 - \int_{[0,\frac{1}{m}]} f \right| \tag{3.6}$$

and, noticing the choice of m and taking $A = [0, \frac{1}{m}]$ in (3.3), one deduces

$$\left| \int_{[0,\frac{1}{m}]} f \right| < \epsilon, \quad \text{for any } f \in \gamma^*. \tag{3.7}$$

Finally, because of (3.6) and (3.7), for any $f \in \gamma^*$, we are led to

$$\left| \int_{[0,1]} \big((f_m - f) g_m\big) \right| > 1 - \epsilon > \lambda,$$

which contradicts (3.5). This completes the proof. □

4. A global optimization problem in the set $\mathcal{D}$

About the densifiability of the set $\mathcal{D}$, defined in (1.4), the main properties are given in the following result.

Lemma 15. *$\mathcal{D}$ is a closed convex precompact set of L_1 for the weak topology, but it is not precompact for the strong topology.*

Proof. The convexity of $\mathcal{D}$ is clear. On the other hand, $\mathcal{D}$ is weakly precompact as consequence of its boundedness [7]. About the closedness of $\mathcal{D}$, consider a sequence (f_n) in $\mathcal{D}$ such that

$$\lim_n f_n = f$$

for the norm topology. Then from the continuity of the norm one has

$$\|f\|_1 = 1. \tag{4.1}$$

Now, since

$$\int_0^1 f = \int_0^1 (f - f_n) + \int_0^1 f_n = \int_0^1 (f - f_n) + 1,$$

by taking the limit as $n \to \infty$, it follows

$$\int_0^1 f = 1.$$

Therefore, from (4.1),

$$\int_0^1 \big(|f| - f\big) = 0$$

and according to

$$|f| - f \geqslant 0,$$

it implies

$$|f| - f = 0 \quad \text{(a.e.)},$$

or equivalently

$$|f| = f \quad \text{(a.e.)}.$$

Hence, we conclude that

$$f \geqslant 0$$

and, again from (4.1),

$$f \in \mathcal{D}.$$

Consequently, $\mathcal{D}$ is a closed set for the strong topology and taking into account the convexity, $\mathcal{D}$ is a closed set for the weak topology [7, p. 130].

On the other hand, noticing L_1 is a Banach for the norm topology, $\mathcal{D}$ is complete. Then, to prove that $\mathcal{D}$ is not precompact for the strong topology it is enough to prove that $\mathcal{D}$ is not compact. Indeed, suppose that $\mathcal{D}$ is compact for the strong topology. Define the sequence

$$f_n \equiv n\chi_{[0,\frac{1}{n}]}, \quad n \in \mathbb{N},$$

where $\chi_{[0,\frac{1}{n}]}$ is the characteristic function of $[0, \frac{1}{n}]$. Then there exists a subsequence $(f_{n_k})_k$ which converges to some $f \in \mathcal{D}$ and one has

$$0 = \lim_{k\to\infty} \int_0^1 |f_{n_k} - f| \geqslant \lim_{k\to\infty} \int_{\frac{1}{n_k}}^1 |f_{n_k} - f| = \lim_{k\to\infty} \int_{\frac{1}{n_k}}^1 |f| = \|f\|_1 = 1,$$

which is a contradiction. Now the lemma follows. □

Since any densifiable set is precompact, from the previous lemma and Corollary 7, it is clear the following result.

Corollary 16. *$\mathcal{D}$ is a densifiable set in L_1 for the weak topology, but it is not densifiable for the norm topology.*

For illustrating the usefulness of the concepts and results obtained until now, we propose the following example.

Example 17. Let $\mathcal{P}$ be the linear functional on the space (L_1, ω_1) defined by

$$\mathcal{P}(f) \equiv \int_0^1 x f(x)\,dx.$$

Thus, the minimization problem

$$\min\{\mathcal{P}(f):\ f \in \mathcal{D}\} \tag{4.2}$$

is not well posed. However, for every weak 0-neighborhood V, the problem (4.2) can be substituted by

$$\min\{\mathcal{P}(f):\ f \in \gamma_V^*\}, \tag{4.3}$$

where γ_V is a V-dense curve in $\mathcal{D}$ and γ_V^* is its image. This new problem has a solution.

Proof. Let us consider in $\mathcal{D}$ the sequence

$$f_n \equiv n\chi_{[0,\frac{1}{n}]}, \quad n = 1, 2, \ldots,$$

with $\chi_{[0,\frac{1}{n}]}$ the characteristic function of $[0, \frac{1}{n}]$, then trivially

$$\mathcal{P}(f_n) = \frac{1}{2n}.$$

Now, by assuming that (4.2) has a solution, say $h \in \mathcal{D}$, it would have

$$\mathcal{P}(h) = 0,$$

and, consequently,

$$h = 0 \quad \text{(a.e.)},$$

which contradicts that $\|h\|_1 = 1$. Therefore, the problem (4.2) does not have any solution, as claimed. However, (4.3) has a solution by virtue of the weak continuity of $\mathcal{P}$, the weak compactness of γ_V^* and the weak densifiability of $\mathcal{D}$. □

When the problem (4.2) has no solution, the problem (4.3) can be interpreted as a sort of algorithm for obtaining approached solutions to the problem (4.2), such as we will see below.

Problem 18. Let θ be the life time of some electronic device, mathematically considered as a random variable belonging to the unit interval $I = [0, 1]$. Assume is given a function $c(\theta, a)$ continuous on the unit square $I \times (0, 1)$, vanishes on the diagonal $\theta = a$ and $c(\theta, a) > 0$ whenever $\theta \neq a$, measuring the error consisting of taking the experimental value a instead of the true value θ. Then the problem consists in knowing if there exists a function $f_0 \in \mathcal{D}$ such that it minimizes the expectation. That is, to wonder if for a fixed value of $a \in (0, 1)$ the problem

$$\min\left\{ \int_0^1 f(\theta)c(\theta, a)\, d\theta\colon\ f \in \mathcal{D} \right\} \tag{M}$$

has a solution.

A study on the problem (M) is given by means of the following result.

Proposition 19. *The problem* (M) *does not have any solution, nevertheless the weak densifiability of* $\mathcal{D}$ *allows us to define an algorithm that approaches a solution of the global optimization problem*

$$f^* \equiv \inf\left\{ \int_0^1 f(\theta)c(\theta, a)\, d\theta\colon\ f \in \mathcal{D} \right\}. \tag{GO}$$

Proof. Since the set

$$\left\{ \int_0^1 f(\theta)c(\theta, a)\, d\theta\colon\ f \in \mathcal{D} \right\}$$

is lower bounded by 0, f^* is a finite and non-negative real number. Since $0 < a < 1$, define for $n > \frac{1}{1-a}$ the function

$$f_{n,a}(\theta) = \begin{cases} n & \text{if } \theta \in [a, a + \frac{1}{n}], \\ 0 & \text{otherwise.} \end{cases}$$

Thus, by the mean value theorem for integrals, there exists θ_n with $a \leqslant \theta_n \leqslant a + \frac{1}{n}$ such that

$$f^* \leqslant \int_0^1 f_{n,a}(\theta)c(\theta, a)\, d\theta = n \int_a^{a+\frac{1}{n}} c(\theta, a)\, d\theta = c(\theta_n, a).$$

Now taking the limit as $n \to \infty$, we deduce that

$$f^* = 0,$$

which implies that the problem (M) has not solution. Otherwise, if there exists $f_0 \in \mathcal{D}$ such that

$$\int_0^1 f_0(\theta)c(\theta, a)\, d\theta = 0,$$

then, necessarily, $f_0 = 0$ (a.e.), which contradicts that

$$\int_0^1 f_0(\theta)\, d\theta = 1.$$

For fixed a, the continuity of $c(\theta, a)$ on I defines a weak 0-neighborhood

$$U_{c,a} \equiv \left\{ f \in L_1 \colon \left| \int_0^1 f(\theta)c(\theta, a)\, d\theta \right| \leqslant 1 \right\}. \tag{4.4}$$

Then, by applying Corollary 7, there exists a curve γ_1, with image γ_1^*, contained in $\mathcal{D}$ such that

$$\mathcal{D} \subset \gamma_1^* + U_{c,a}.$$

Now, the problem (M) on the compact γ_1^* has a solution, that is, there exists a function $f_1 \in \gamma_1^*$ such that

$$\min\left\{ \int_0^1 f(\theta)c(\theta, a)\, d\theta \colon\ f \in \gamma_1^* \right\} = \int_0^1 f_1(\theta)c(\theta, a)\, d\theta \equiv f_1^*.$$

Analogously, for each $n \geqslant 2$, by considering the neighborhood $\frac{1}{n}U_{c,a}$, there exists a curve γ_n in $\mathcal{D}$, with image denoted by γ_n^*, such that

$$\mathcal{D} \subset \gamma_n^* + \frac{1}{n}U_{c,a}. \tag{4.5}$$

By compactness the problem (M) has a solution on γ_n^* and so there exists $f_n \in \gamma_n^*$ such that

$$\min\left\{ \int_0^1 f(\theta)c(\theta, a)\, d\theta \colon\ f \in \gamma_n^* \right\} = \int_0^1 f_n(\theta)c(\theta, a)\, d\theta \equiv f_n^*. \tag{4.6}$$

Now, we claim that the algorithm is convergent, i.e.,

$$\lim_n f_n^* = f^*.$$

Indeed, we consider the functional, denoted by $\mathcal{P}_{c,a}$, defined on L_1 by the function $c(\theta, a)$,

$$\mathcal{P}_{c,a}(f) \equiv \int_0^1 f(\theta)c(\theta, a)\, d\theta, \quad f \in L_1.$$

Since $f^* = 0$, given an arbitrary $\epsilon > 0$, determine a function $g \in \mathcal{D}$ such that

$$\mathcal{P}_{c,a}(g) < \frac{\epsilon}{2}$$

and a positive integer $m > \frac{2}{\epsilon}$. Because of (4.5), given $g \in \mathcal{D}$ for each $n \geqslant m$ there exists a function $g_n \in \gamma_n^*$ such that

$$g - g_n \in \frac{1}{n}U_{c,a}.$$

Noticing (4.4) and (4.6), for any $n \geqslant m$, we can write

$$\frac{\epsilon}{2} > \mathcal{P}_{c,a}(g) = \mathcal{P}_{c,a}(g - g_n) + \mathcal{P}_{c,a}(g_n) \geqslant -\frac{1}{n} + \mathcal{P}_{c,a}(f_n) = -\frac{1}{n} + f_n^* > -\frac{\epsilon}{2} + f_n^*,$$

which is equivalent to

$$0 \leqslant f_n^* < \epsilon \quad \text{for all } n \geqslant m$$

and, in consequence,

$$\lim_n f_n^* = f^*,$$

as claimed. Now, the proof is completed. □

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